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Third-Order Asymptotic Properties of a Class of Test Statistics under a Local Alternative

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Suppose that $\{X_i; i=1, 2, ...,\}$ is a sequence of p-dimensional random vectors forming a stochastic process. Let $p_{n,\theta}(\mathbf{x}_n)$, $\mathbf{x}_n \in \mathbb{R}^{np}$, be the probability density function of $\mathbf{X}_n = (X_1, ..., X_n)$ depending on $\theta \in \Theta$, where Θ is an open set of $\mathbb{R}^1$. We consider to test a simple hypothesis $H: \theta = \theta_0$ against the alternative $A: \theta \neq \theta_0$. For this testing problem we introduce a class of tests $\mathscr{S}$, which contains the likelihood ratio, Wald, modified Wald, and Rao tests as special cases. Then we derive the third-order asymptotic expansion of the distribution of $T \in \mathscr{S}$ under a sequence of local alternatives. Using this result we elucidate various third-order asymptotic properties of $T \in \mathscr{S}$ (e.g., Bartlett's adjustments, third-order asymptotically most powerful properties). Our results are very general, and can be applied to the i.i.d. case, multivariate analysis, and time series analysis. Two concrete examples will be given. One is a Gaussian ARMA process (dependent case), and the other is a non-linear regression model (non-identically distributed case).

1. INTRODUCTION

Suppose that $\mathbf{X}_n = (X_1, ..., X_n)$ is a collection of p-dimensional random vectors forming a stochastic process. Let $p_{n,\theta}(\mathbf{x}_n)$, $\mathbf{x}_n \in \mathbb{R}^{np}$, be the probability density function of $\mathbf{X}_n$ depending on $\theta \in \Theta$, where Θ is an open set of $\mathbb{R}^1$. The problem considered is that of testing a simple hypothesis $H: \theta = \theta_0$ against the alternative $A: \theta \neq \theta_0$. For this problem we propose a class of tests $\mathscr{S}$, which contains the likelihood ratio (LR), Wald (W), modified Wald (MW), and Rao (R) tests as special cases. Then we derive the third-order asymptotic expansion of the distribution of $T \in \mathscr{S}$ under a sequence of local alternatives $\theta = \theta_0 + \varepsilon/c_n$, where $\varepsilon > 0$ and $\{c_n\}$ is an appropriate

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sequence of positive numbers satisfying $c_n \to \infty$ as $n \to \infty$. The resulting automatic formula is very general, hence it can be applied to the i.i.d. case, multivariate analysis, and time series analysis. Using this formula we elucidate various third-order asymptotic properties of $T \in \mathscr{S}$.

In Section 3 we discuss Bartlett's adjustment procedure. Since $T \in \mathscr{S}$ is not generally adjustable in the sense of Bartlett (e.g., Taniguchi [14]) we give a sufficient condition that a modified test $T^* = h(\hat{\theta}_{\mathrm{ML}})\, T$ is adjustable in the sense of Bartlett, where $h(\theta)$ is a smooth function and $\hat{\theta}_{\mathrm{ML}}$ is the maximum likelihood estimator of θ_0. Under this sufficient condition we can apply the Bartlett adjustment to T^*. The resulting test statistic is written in the form $T^{**} = (1 + c_n^{-2} \rho^*)\, T^*$, where ρ^* is the Bartlett adjustment factor. Then we give the third-order asymptotic expansion of the distribution of T^{**} under a sequence of local alternatives $\theta = \theta_0 + \varepsilon/c_n$. This result implies that the second-order asymptotic powers of all the modified tests T^{**} are equal, and that there is in general no test in $\mathscr{S}$ which is third-order asymptotically most powerful uniformly in ε.

In Section 4 it is shown that we can find the third-order asymptotically most powerful test in $\mathscr{S}$ at any specified $\varepsilon > 0$ and level x. Two concrete examples will be given. One is a Gaussian ARMA process (dependent case), and the other is a nonlinear regression model (non-identically distributed case).

Throughout this paper we restrict ourselves to the situation where θ is scalar because extension of our results to the multiparameter case causes long and complex formulae (although it is methodologically straightforward) and obscures our theoretical framework.

2. General Theory

In this section, for a testing problem, we introduce a class of tests $\mathscr{S}$, and derive the third-order asymptotic expansion of the distribution of $T \in \mathscr{S}$ under a sequence of local alternatives. This result can be applied to the i.i.d. case, multivariate analysis, and time series analysis.

We require the following assumptions:

(1) $p_{n,\theta}(\mathbf{x}_n)$ is continuously five times differentiable with respect to $\theta \in \Theta$.

(2) The partial derivative $\partial/\partial\theta$ and the expectation E_θ with respect to $p_{n,\theta}(\mathbf{x}_n)$ are interchangeable.

(3) For an appropriate sequence $\{c_n\}$ satisfying $c_n \to \infty$ as $n \to \infty$, the asymptotic cumulants of

$$Z_i(\theta) = c_n^{-1} \left\{ \frac{\partial^i}{\partial\theta^i} \log p_{n,\theta}(\mathbf{X}_n) - E_\theta \frac{\partial^i}{\partial\theta^i} \log p_{n,\theta}(\mathbf{X}_n) \right\} \qquad (i = 1, 2, 3),$$

possess asymptotic expansions of the form

$$\mathrm{cum}_\theta\{Z_i(\theta), Z_j(\theta)\} = \kappa_{ij}^{(1)}(\theta) + c_n^{-2}\kappa_{ij}^{(2)}(\theta) + o(c_n^{-2}), \tag{2.1}$$

$$c_\theta\{Z_i(\theta), Z_j(\theta), Z_k(\theta)\} = c_n^{-1}\kappa_{ijk}^{(1)}(\theta) + o(c_n^{-2}), \tag{2.2}$$

$$\mathrm{cum}_\theta\{Z_i(\theta), Z_j(\theta), Z_k(\theta), Z_m(\theta)\} = c_n^{-2}\kappa_{ijkm}^{(1)}(\theta) + o(c_n^{-2}), \tag{2.3}$$

$i, j, k, m = 1, 2, 3$, and the Jth-order ($J \geqslant 5$) cumulants satisfy

$$\mathrm{cum}_\theta^{(J)}\{Z_{i_1}(\theta), ..., Z_{i_J}(\theta)\} = O(c_n^{-J+2}), \tag{2.4}$$

where $i_1, ..., i_J \in \{1, 2, 3\}$.

Henceforth we adopt the following notations: $I(\theta) = \kappa_{11}^{(1)}(\theta)$, $J(\theta) = \kappa_{12}^{(1)}(\theta)$, $L(\theta) = \kappa_{13}^{(1)}(\theta)$, $M(\theta) = \kappa_{22}^{(1)}(\theta)$, $K(\theta) = \kappa_{111}^{(1)}(\theta)$, $N(\theta) = \kappa_{112}^{(1)}(\theta)$, $H(\theta) = \kappa_{1111}^{(1)}(\theta)$, and $\Delta(\theta) = \kappa_{11}^{(2)}(\theta)$, because the resulting formula in the asymptotic expansion can be expressed solely in terms of them. Occasionally we shall use the simpler notations Z_i, I, J, L, etc. instead of $Z_i(\theta)$, $I(\theta)$, $J(\theta)$, $L(\theta)$, etc.

Consider the transformation

$$W_1 = Z_1/\sqrt{I},$$
$$W_2 = Z_2 - J \cdot I^{-1} Z_1,$$
$$W_3 = Z_3 - L \cdot I^{-1} Z_1.$$

For the testing problem $H: \theta = \theta_0$ against $A: \theta \neq \theta_0$, we introduce the following class of tests:

$$\begin{aligned}\mathscr{S} = \{T \mid T = {} & W_1^2 + c_n^{-1}(a_1 W_1^2 W_2 + a_2 W_1^3) \\ & + c_n^{-2}(b_1 W_1^2 + b_2 W_1^2 W_2^2 + b_3 W_1^4 + b_4 W_1^3 W_2 + b_5 W_1^3 W_3) + o_p(c_n^{-2}), \\ & \text{under } H, \text{ where } a_i\,(i = 1, 2) \text{ and} \\ & b_i\,(i = 1, ..., 5) \text{ are nonrandom constants}\}.\end{aligned}$$

This class $\mathscr{S}$ is a very natural one. We can show that four famous tests below belong to $\mathscr{S}$ in the same way as Taniguchi [14] did for Gaussian ARMA processes.

Examples. Let $\hat{\theta}_{\mathrm{ML}}$ be the maximum likelihood estimator of θ_0, and put $l_n(\theta) = \log p_{n,\theta}(\mathbf{X}_n)$.

(i) The likelihood ratio test $\mathrm{LR} = 2[l_n(\hat{\theta}_{\mathrm{ML}}) - l_n(\theta_0)]$ belongs to $\mathscr{S}$ with the coefficients $a_1 = 1/I$, $a_2 = -K/3I^{3/2}$, $b_1 = -\Delta/I$, $b_2 = 1/I^2$, $b_3 = (J+K)^2/4I^3 - (3M + 6N + H)/12I^2$, $b_4 = -(J+K)/I^{5/2}$, and $b_5 = 1/3I^{3/2}$.

(ii) Wald's test $W = c_n^2(\hat{\theta}_{\mathrm{ML}} - \theta_0)^2 I(\hat{\theta}_{\mathrm{ML}})$ belongs to $\mathscr{S}$ with the coefficients $a_1 = 2/I$, $a_2 = J/I^{3/2}$, $b_1 = -2\Delta/I$, $b_2 = 3/I^2$, $b_3 = -(3J^2 + 4JK + K^2)/4I^3 + (4L + 3N + H)/6I^2$, $b_4 = -K/I^{5/2}$, and $b_5 = 1/I^{3/2}$.

(iii) A modified Wald's test $\mathrm{MW} = c_n^2(\hat{\theta}_{\mathrm{ML}} - \theta_0)^2 I(\theta_0)$ belongs to $\mathscr{S}$ with the coefficients $a_1 = 2/I$, $a_2 = -(J+K)/I^{3/2}$, $b_1 = -2\Delta/I$, $b_2 = 3/I^2$, $b_3 = (9J^2 + 14JK + 5K^2)/4I^3 - (L + 3M + 6N + H)/3I^2$, $b_4 = -(6J + 4K)/I^{5/2}$, and $b_5 = 1/I^{3/2}$.

(iv) Rao's test $\mathrm{R} = Z_1(\theta_0)^2 I(\theta_0)^{-1}$ belongs to $\mathscr{S}$ with the coefficients $a_1 = a_2 = b_1 = b_2 = b_3 = b_4 = b_5 = 0$.

Now we proceed to derive the third-order asymptotic expansion of the distribution of $T \in \mathscr{S}$ under a sequence of local alternatives $\theta = \theta_0 + \varepsilon/c_n$, $(\varepsilon > 0)$. Since the actual calculation procedure is formidable we give a sketch of the derivation. First, we evaluate the characteristic function of T, i.e.,

$$\psi_n(t, \varepsilon) = E_{\theta_0 + \varepsilon/c_n} e^{itT}, \qquad T \in \mathscr{S}.$$

Denoting $L_n(\mathbf{x}_n) = p_{n,\theta_0 + \varepsilon/c_n}(\mathbf{x}_n)/p_{n,\theta_0}(\mathbf{x}_n)$, we have

$$\begin{aligned} \psi_n(t, \varepsilon) &= \int e^{itT(\mathbf{x}_n)} L_n(\mathbf{x}_n)\, p_{n,\theta_0}(\mathbf{x}_n)\, d\mathbf{x}_n \\ &= E_{\theta_0} e^{\{itT + \log L_n\}}. \end{aligned} \tag{2.5}$$

We expand $\log L_n(\mathbf{X}_n)$ in a Taylor series in ε/c_n, leading to

$$\begin{aligned} \log L_n(\mathbf{X}_n) &= \varepsilon\sqrt{I}\, W_1 - \frac{I\varepsilon^2}{2} + c_n^{-1}\left\{\frac{\varepsilon^2 J}{2\sqrt{I}} W_1 + \frac{\varepsilon^2}{2} W_2 - \frac{\varepsilon^3}{6}(3J + K)\right\} \\ &\quad + c_n^{-2}\left\{\frac{\varepsilon^2 L}{6\sqrt{I}} W_1 + \frac{\varepsilon^3}{6} W_3 - \frac{\Delta\varepsilon^2}{2} - \frac{\varepsilon^4}{24}(4L + 3M + 6N + H)\right\} \\ &\quad + o_p(c_n^{-2}). \end{aligned} \tag{2.6}$$

Inserting (2.6) in $e^{\{itT + \log L_n\}}$ we obtain, after further expansion and collection of terms,

$$\begin{aligned} e^{\{itT + \log L_n\}} &= e^{\{itW_1^2 + \varepsilon\sqrt{I} W_1 - I\varepsilon^2/2\}} \times \left\{1 + \frac{1}{c_n} q_1(W_1, W_2)\right. \\ &\quad \left. + \frac{1}{c_n^2} q_2(W_1, W_2, W_3) + o_p(c_n^{-2})\right\}, \end{aligned} \tag{2.7}$$

where $q_1(\ ,\ )$ and $q_2(\ ,\ ,\ )$ are polynomials. In view of the assumption

(3) we can easily evaluate the asymptotic cumulants of $\mathbf{W} = (W_1, W_2, W_3)'$. Thus the third-order Edgeworth expansion of the distribution of $\mathbf{W}$ is given in the form (see Taniguchi [12, (3.7)]),

$$\begin{aligned}
&P_{n,\theta_0}[W_1 < w_1, W_2 < w_2, W_3 < w_3] \\
&= \int_{-\infty}^{w_1} \int_{-\infty}^{w_2} \int_{-\infty}^{w_3} f_1(w_1) f_2(w_2, w_3) \\
&\quad \times \left[1 + \frac{1}{6c_n} \sum_{j,k,l=1}^{3} c_{jkl}^{(1)} H_{jkl}(\mathbf{w}) + \frac{1}{2c_n^2} \sum_{j,k=1}^{3} c_{jk}^{(3)} H_{jk}(\mathbf{w}) \right. \\
&\quad + \frac{1}{24c_n^2} \sum_{j,k,l,m=1}^{3} c_{jklm}^{(1)} H_{jklm}(\mathbf{w}) \\
&\quad \left. + \frac{1}{72c_n^2} \sum_{\substack{j,k,l,\\ j',k',l'=1}} c_{jkl}^{(1)} c_{j'k'l'}^{(1)} H_{jklj'k'l'}(\mathbf{w}) \right] d\mathbf{w} + o(c_n^{-2}) \\
&= \int_{-\infty}^{w_1} \int_{-\infty}^{w_2} \int_{-\infty}^{w_3} q(\mathbf{w})\, d\mathbf{w} + o(c_n^{-2}), \qquad \text{say,}
\end{aligned} \tag{2.8}$$

where $\mathbf{w} = (w_1, w_2, w_3)'$, $f_1(w_1) = (2\pi)^{-1/2} \exp - w_1^2/2$, $f_2(w_2, w_3) = (2\pi)^{-1} |\Omega_2|^{-1/2} \exp - \frac{1}{2}(w_2, w_3)\, \Omega_2^{-1} (w_2, w_3)'$, and $H_{j_1 \cdots j_s}(\mathbf{w})$ are the Hermite polynomials. Here the above coefficients $c_{\cdot\cdot\cdot}^{(\cdot)}$ and the matrix Ω_2 can be expressed in terms of the cumulants $\kappa_{\cdot\cdot\cdot}^{(\cdot)}$ given in (3). From (2.5), (2.7), and (2.8) it follows that

$$\begin{aligned}
\psi_n(t, \varepsilon) = \iiint e^{\{itw_1^2 + \varepsilon\sqrt{I} w_1 - I\varepsilon^2/2\}} \times [1 + c_n^{-1} q_1(w_1, w_2) \\
+ c_n^{-2} q_2(w_1, w_2, w_3)] \times q(\mathbf{w})\, d\mathbf{w} + o(c_n^{-2}).
\end{aligned} \tag{2.9}$$

First, we calculate the integral (2.9) with respect to w_2 and w_3. Then, integration with respect to w_1 yields

$$\begin{aligned}
\psi_n(t, \varepsilon) = \exp\left\{ \frac{\varepsilon^2 Iit}{1 - 2it} \right\} \times (1 - 2it)^{-1/2} \\
\times \left[1 + c_n^{-1} \sum_{j=0}^{3} B_j^{(T)} (1 - 2it)^{-j} \right. \\
\left. + c_n^{-2} \sum_{j=0}^{6} C_j^{(T)} (1 - 2it)^{-j} \right] + o(c_n^{-2}),
\end{aligned} \tag{2.10}$$

where

$$B_0^{(T)} = -(3J+K)\,\varepsilon^3/6, \qquad B_1^{(T)} = J\varepsilon^3/2 - (K + 3I^{3/2}a_2)\,\varepsilon/2I,$$

$$B_2^{(T)} = -I^{3/2}a_2\varepsilon^3/2 + (K + 3I^{3/2}a_2)\,\varepsilon/2I,$$

$$B_3^{(T)} = (K + 3I^{3/2}a_2)\,\varepsilon^3/6,$$

$$C_6^{(T)} = (K + 3a_2 I^{3/2})^2\,\varepsilon^6/72,$$

$$C_5^{(T)} = 5(3a_2 I^{3/2} + K)^2\,\varepsilon^4/24I - a_2 I^{3/2}(3a_2 I^{3/2} + K)\,\varepsilon^6/12,$$

$$\begin{aligned} C_4^{(T)} = {} & (3a_2^2 I^3 + 6a_2 J I^{3/2} + 2KJ)\,\varepsilon^6/24 \\ & + (3\tilde{M}a_1^2 I^3 + 6\tilde{N}a_1 I^2 - 90a_2^2 I^3 - 36Ka_2 I^{3/2} \\ & + 12b_3 I^3 + HI - 5K^2)\,\varepsilon^4/24I + 5(K + 3I^{3/2}a_2)^2\,\varepsilon^2/8I^2, \end{aligned}$$

$$\begin{aligned} C_3^{(T)} = {} & \{-3a_2(6J+K)\,I^{3/2} - K(3J+K)\}\,\varepsilon^6/36 \\ & + \{-4b_3 I^3 - 2\tilde{M}a_1^2 I^3 - 2\tilde{N}a_1 I^2 + 15a_2^2 I^3 \\ & + 2a_2(K+6J)\,I^{3/2} + 4JK\}\,\varepsilon^4/8I \\ & + (12b_3 I^3 + 3\tilde{M}a_1^2 I^3 + 6\tilde{N}a_1 I^2 - 45a_2^2 I^3 - 21Ka_2 I^{3/2} \\ & + HI - 5K^2)\,\varepsilon^2/4I^2 + 5(K + 3a_2 I^{3/2})^2/24I^3, \end{aligned}$$

$$\begin{aligned} C_2^{(T)} = {} & \{3J^2 + 2a_2 I^{3/2}(3J+K)\,\varepsilon^6/24 \\ & + \{3\tilde{M}a_1^2 I^3 + 6\tilde{M}a_1 I^2 - 6a_2 I^{3/2}(9J+K) + 6\tilde{N}I \\ & - 2K(6J+K)\}\,\varepsilon^4/24I \\ & + \{4b_1 I^3 + 4b_2\tilde{M}I^3 - 24b_3 I^3 - 12\tilde{M}a_1^2 I^3 - 14\tilde{N}a_1 I^2 + 45a_2^2 I^3 \\ & + 2(3J+6K)\,I^{3/2}\,a_2 + 4I^2\Delta - 2HI + 2KJ + 5K^2\}\,\varepsilon^2/8I^2 \\ & + \{3\tilde{M}a_1^2 I^3 + 6\tilde{N}a_1 I^2 - 30a_2^2 I^3 \\ & - 16Ka_2 I^{3/2} + 12b_3 I^3 + HI - 5K^2\}/8I^3, \end{aligned}$$

$$\begin{aligned} C_1^{(T)} = {} & -J(3J+K)\,\varepsilon^6/12 \\ & + \{4LI + 3J^2 - 6\tilde{M}a_1 I^2 + 2(3a_2 I^{3/2} + K)(3J+K)\,\varepsilon^4/24I \\ & + \{-2b_1 I^3 - 2b_2\tilde{M}I^3 + \tilde{N}I - KJ + 3\tilde{M}a_1^2 I^3 \\ & + (\tilde{M} + \tilde{N})\,a_1 I^2 - 3a_2 J I^{3/2}\}\,\varepsilon^2/4I^2 + \{-6\tilde{M}a_1^2 I^3 \\ & - 8\tilde{N}a_1 I^2 + 15a_2^2 I^3 + 6Ka_2 I^{3/2} + 4b_1 I^3 \\ & + 4\tilde{M}b_2 I^3 - 12b_3 I^3 + 4\Delta I^2 - 2HI + 5K^2\}/8I^3, \end{aligned}$$

$$\begin{aligned} C_0^{(T)} = {} & (3J+K)^2\,\varepsilon^6/72 + (3\tilde{M} - 4L - 3M - 6N - H)\,\varepsilon^4/24 \\ & + (-2\Delta I - \tilde{M}a_1 I - \tilde{N})\varepsilon^2/4I + (-12b_1 I^3 \\ & - 12\tilde{M}b_2 I^3 + 9\tilde{M}a_1^2 I^3 + 6\tilde{N}a_1 I^2 - 12\Delta I^2 + 3HI - 5K^2)/24I^3, \end{aligned}$$

with $\tilde{M}=M-J^2/I$ and $\tilde{N}=N-JKI^{-1}$. Inverting (2.10) by the Fourier inverse transform we have

THEOREM 1. *The distribution function of $T \in \mathscr{S}$ under a sequence of local alternatives $\theta=\theta_0+\varepsilon/c_n$ has the asymptotic expansion*

$$
\begin{aligned}
P_{n,\theta_0+\varepsilon/c_n}&[T \leqslant x] \\
&= P[\chi_1^2(\delta) \leqslant x] + c_n^{-1} \sum_{j=0}^{3} B_j^{(T)} P[\chi_{1+2j}^2(\delta) \leqslant x] \\
&\quad + c_n^{-2} \sum_{j=0}^{6} C_j^{(T)} P[\chi_{1+2j}^2(\delta) \leqslant x] + o(c_n^{-2}), \qquad (2.11)
\end{aligned}
$$

where $\delta^2 = I\varepsilon^2/2$, and $\chi_j^2(\delta)$ is a noncentral χ^2 random variable with j degrees of freedom and noncentrality parameter δ^2.

Theorem 1 can be applied to (i) dependent observations, and (ii) not identically distributed observations. Concrete examples of (i) and (ii) will be given in Section 4. We end this section with the following remark.

Remark 1. For a random sample from a multivariate normal distribution, Sugiura [9] considered the problem of testing the equality of a covariance matrix ($=\sum$) to a given matrix ($=\sum_0$). Then he gave the third-order asymptotic expansion of the LR statistic under a sequence of local alternatives $A_n: \sum_0^{-1/2} \sum \sum_0^{-1/2} = I + n^{-1/2}\Theta$, where Θ is a symmetric matrix. Our formula (2.11) for this problem agrees with that of Sugiura [9] when the parameter concerned is scalar.

3. BARTLETT ADJUSTMENTS

Bartlett's adjustment procedure has been elucidated in various directions. Barndorff-Nielsen and Cox [2] established a simple connection between the Bartlett adjustment factor of the likelihood ratio statistic LR and the normalizing constant for the density of a maximum likelihood estimator conditioned on an ancillary statistic. They discussed various expressions for these quantities. Barndorff-Nielsen and Blaesild [3] described, for the numerical calculation of Bartlett adjustments of LR, a method which may be of use when the cumulants of the log likelihood derivatives are easy to determine in one parametrization while the hypotheses to be tested are all linear in some other parametrization. For Gaussian ARMA processes Taniguchi [14] gave a necessary and sufficient condition for $T \in \mathscr{S}$ such that T is adjustable in the sense of Bartlett, and showed that, among the four tests LR, W, MW, and R, the LR test is the only one which satisfies this condition.

Now we explain Bartlett's adjustment in our situation. Under the null hypothesis H it is easy to see that the expectation of $T \in \mathscr{S}$ can be written as

$$E(T) = 1 - \rho/c_n^2 + o(c_n^{-2}),$$

and that

$$T/E\{T\} = \left(1 + \frac{\rho}{c_n^2}\right) T + o_p(c_n^{-2}).$$

Henceforth ρ is called the Bartlett adjustment factor. If the terms of order c_n^{-2} in the asymptotic expansion of the distribution of $T^* = (1 + \rho/c_n^2)\, T$ vanish (i.e., $P_{n,\theta_0}[T^* \leqslant x] = P[\chi_1^2 \leqslant x] + o(c_n^{-2})$), we say that T is adjustable in the sense of Bartlett (B-adjustable for short).

In view of Taniguchi [14], $T \in \mathscr{S}$ is not generally B-adjustable. Thus we consider modification of T to $T^* = h(\hat{\theta}_{\mathrm{ML}})\, T$ so that T^* is B-adjustable, where $h(\theta)$ is a smooth function. Then we can state

THEOREM 2. *Suppose that $h(\theta)$ is continuously three times differentiable. For $T \in S$, the modified test $T^* = h(\hat{\theta}_{\mathrm{ML}})\, T$ is B-adjustable if $h = h(\theta_0)$, $h' = h'(\theta_0)$ and $h'' = h''(\theta_0)$ satisfy*

(i) $h = 1$,

(ii) $h' = -(3a_2 I^{3/2} + K)/3I$,

(iii) $h'' = -I\tilde{M}a_1^2/2 - \tilde{N}a_1 - 2Ib_3 + (2I^{3/2}a_2 - J - K)(3a_2 I^{3/2} + K)/3I^2 - (IH - 3K^2)/6I^2$.

Proof. We give a sketch of the proof because the actual calculation is troublesome. The stochastic expansion of $\hat{\theta}_{\mathrm{ML}}$ is given by

$$\hat{\theta}_{\mathrm{ML}} - \theta_0 = c_n^{-1} I^{-1/2} W_1 + c_n^{-2}\{I^{-3/2} W_1 W_2 - (J + K)\, W_1^2/2I^2\} + o_p(c_n^{-2}), \tag{3.1}$$

(see Taniguchi [12–14]). Expand T^* as

$$\begin{aligned} T^* &= h(\hat{\theta}_{\mathrm{ML}})\, T \\ &= \{1 + (\hat{\theta}_{\mathrm{ML}} - \theta_0)\, h'(\theta_0) + \tfrac{1}{2}(\hat{\theta}_{\mathrm{ML}} - \theta_0)^2\, h''(\theta_0)\}\, T + o_p(c_n^{-2}). \end{aligned} \tag{3.2}$$

Inserting (3.1) in (3.2) we obtain

$$\begin{aligned} T^* = {} & W_1^2 + c_n^{-1}[a_1 W_1^2 W_2 + a_2' W_1^3] \\ & + c_n^{-2}[b_1 W_1^2 + b_2 W_1^2 W_2^2 + b_3' W_1^4 + b_4' W_1^3 W_2 + b_5 W_1^3 W_3] + o_p(c_n^{-2}), \end{aligned} \tag{3.3}$$

where $a_2' = a_2 + h'I^{-1/2}$, $b_3' = b_3 + h'a_2 I^{-1/2} - (J+K)\, h'/2I^2 + h''/2I$, and $b_4' = b_4 + h'a_1 I^{-1/2} + h'I^{-3/2}$. This implies $T^* \in \mathscr{S}$, and hence a necessary and sufficient condition for its Bartlett adjustability is that the coefficients satisfy

(1) $a_2' = -K/3I^{3/2}$,

(2) $3I^3 \tilde{M} a_1^2 + 6I^2 \tilde{N} a_1 + 12 I^3 b_3' + IH - 3K^2 = 0$

(see Theorem 2 of Taniguchi [14]). Solving (1) and (2) with respect to h' and h'', we obtain the relations (ii) and (iii). ∎

For $h(\theta)$ satisfying (i), (ii), and (iii), by evaluating the expectation of (3.3) we find that the Bartlett adjustment factor ρ^* of T^* is given by

$$\begin{aligned}\rho^* = -\{&12I^2\Delta + 12I^3 b_1 + 12I^3 \tilde{M} b_2 \\ &- 9I^3 \tilde{M} a_1^2 - 6I^2 \tilde{N} a_1 - 3IH + 5K^2\}/12I^3. \end{aligned} \tag{3.4}$$

It follows that the stochastic expansion of $T^{**} = (1 + \rho^*/c_n^2)\, T^*$ is

$$\begin{aligned} T^{**} &= T^* + \frac{\rho^*}{c_n^2} W_1^2 + o_p(c_n^{-2}) \\ &= W_1^2 + c_n^{-1}[a_1 W_1^2 W_2 + a_2' W_1^3] + c_n^{-2}[b_1'' W_1^2 + b_2 W_1^2 W_2^2 \\ &\quad + b_3' W_1^4 + b_4' W_1^3 W_2 + b_5 W_1^3 W_3] + o_p(c_n^{-2}), \end{aligned} \tag{3.5}$$

where

$$b_1'' = -(12I^2\Delta + 12I^3 \tilde{M} b_2 - 9I^3 \tilde{M} a_1^2 - 6I^2 \tilde{N} a_1 - 3IH + 5K^2)/12I^3.$$

Application of Theorem 1 to (3.5) leads to

THEOREM 3. *Suppose that $h(\theta)$ satisfies* (i)–(iii) *in Theorem* 2, *and that ρ^* is given by* (3.4). *Then, for $T \in \mathscr{S}$, the distribution function of the modified test $T^{**} = (1 + c_n^{-2}\rho^*)\, h(\hat{\theta}_{\mathrm{ML}})\, T$ under a sequence of local alternatives $\theta = \theta_0 + \varepsilon/c_n$, has the third-order asymptotic expansion*

$$\begin{aligned} &P_{n,\theta_0+\varepsilon/c_n}[T^{**} \leqslant x] \\ &\quad = P[\chi_1^2(\delta) \leqslant x] + c_n^{-1} \sum_{j=0}^{2} B_j^{(T^{**})} P[\chi_{1+2j}^2(\delta) \leqslant x] \\ &\qquad + c_n^{-2} \sum_{j=0}^{4} C_j^{(T^{**})} P[\chi_{1+2j}^2(\delta) \leqslant x] + o(c_n^{-2}), \end{aligned} \tag{3.6}$$

where

$$B_0^{(T^{**})} = -(3J+K)\,\varepsilon^3/6, \qquad B_1^{(T^{**})} = J\varepsilon^3/2, \qquad B_2^{(T^{**})} = K\varepsilon^3/6,$$

$$C_4^{(T^{**})} = K^2\varepsilon^6/72, \qquad C_3^{(T^{**})} = JK\varepsilon^6/12 + (-3\tilde{M}I^2a_1^2 + H)\,\varepsilon^4/24,$$

$$\begin{aligned} C_2^{(T^{**})} = {} & (9J^2 - 6JK - 2K^2)\,\varepsilon^6/72 \\ & + (\tilde{M}I^3a_1^2 + 2\tilde{M}I^2a_1 + 2KJ + 2\tilde{N}I)\,\varepsilon^4/8I \\ & + (-9\tilde{M}I^3a_1^2 + 3HI - 5K^2)\,\varepsilon^2/24I^2, \end{aligned}$$

$$\begin{aligned} C_1^{(T^{**})} = {} & -J(3J+K)\,\varepsilon^6/12 + (4IL + 3J^2 - 6\tilde{M}I^2a_1)\,\varepsilon^4/24I \\ & + (-3HI + 12I^2\Delta + 5K^2 + 6\tilde{N}I + 9\tilde{M}I^3a_1^2 + 6\tilde{M}I^2a_1)\,\varepsilon^2/24I^2, \end{aligned}$$

$$\begin{aligned} C_0^{(T^{**})} = {} & (3J+K)^2\,\varepsilon^6/72 + (3\tilde{M} - 4L - 3M - 6N - H)\,\varepsilon^4/24 \\ & + (-2I\Delta - \tilde{N} - \tilde{M}Ia_1)\,\varepsilon^2/4I. \end{aligned}$$

In this theorem we observe that the coefficients $B^{(\cdot)}_{\cdot}$ in (3.6) are independent of $T \in \mathscr{S}$, and hence all the powers of the modified tests T^{**} are identical up to second-order. On the other hand, the coefficients $C^{(\cdot)}_{\cdot}$ in (3.6) depend on $T \in \mathscr{S}$ unless $\tilde{M} = 0$. Therefore, in general, there is no test which is third-order uniformly most powerful in $\mathscr{S}$ unless $\tilde{M} = 0$. Here we note that $\gamma_\theta = \tilde{M}^{1/2}/I$ is a counterpart of Efron's statistical curvature in time series analysis. Therefore the results above agree with those of Kumon and Amari [7] and Amari [1] which elucidate higher-order asymptotics of tests for a curved exponential family. In the next section we will give a further study of third-order asymptotics of tests.

4. Third-Order Asymptotics of Tests and Examples

In the previous section we saw that the third-order terms of the power of the modified test T^{**} depend on T unless $\tilde{M} = 0$. However, in this section, it is shown that we can find a third-order optimal test in $\mathscr{S}$ at each fixed ε and level x.

The relation

$$P[\chi^2_{j+2}(\delta) > x] - P[\chi^2_j(\delta) > x] = 2p_{j+2}(x;\delta^2) \tag{4.1}$$

is well known, where $p_j(x;\delta^2)$ is the probability density function of $\chi^2_j(\delta)$. From (4.1) and Theorem 3, the power of T^{**} up to third-order can be written as

$$\begin{aligned} P_{n,\theta_0+\varepsilon/c_n}[T^{**}>x] &= P[\chi_1^2(\delta)>x] + c_n^{-1} \sum_{j=0}^{2} B_j^{(T^{**})} P[\chi_{1+2j}^2(\delta)>x] \\ &\quad + 2c_n^{-2}[A_4 p_9(x;\delta^2) + \{A_3 - \tilde{M}I^2 a_1^2 \varepsilon^4/8\} p_7(x;\delta^2) \\ &\quad + \{A_2 + \tilde{M}Ia_1\varepsilon^4/4 - 3\tilde{M}Ia_1^2\varepsilon^2/8\} p_5(x;\delta^2) \\ &\quad + \{A_1 + \tilde{M}a_1\varepsilon^2/4\} p_3(x;\delta^2)] + o(c_n^{-2}), \end{aligned} \tag{4.2}$$

where $A_1, ..., A_4$ depend only on θ_0 and ε, i.e., are independent of a_1. Differentiating the third-order terms of (4.2) with respect to a_1, we can see that for each given ε and x,

$$\begin{aligned} a_1 = a_1(\varepsilon, x, \theta_0) = \{&I\varepsilon^2 p_5(x;\delta^2) \\ &+ p_3(x;\delta^2)\}/\{I^2\varepsilon^2 p_7(x;\delta^2) + 3Ip_5(x;\delta^2)\} \end{aligned} \tag{4.3}$$

gives the maximum of the third-order terms with respect to a_1. Hence,

THEOREM 4. *Let ε and x be a given positive number and a given level, respectively. Suppose that a test T belongs to $\mathscr{S}$ with the coefficient $a_1 = a_1(\varepsilon, x, \theta_0)$ given by* (4.3). *Then the modified test T^{**} is third-order asymptotically most powerful in the sense that it maximizes the power* (4.2) *up to third order.*

Setting $\varepsilon = I^{-1/2}t$, we rewrite (4.3) as follows:

$$\begin{aligned} Ia_1 &= \{t^2 p_5(x; t^2/2) + p_3(x; t^2/2)\}/\{t^2 p_7(x; t^2/2) + 3p_5(x; t^2/2)\} \\ &= f(t), \qquad \text{say.} \end{aligned}$$

Let x_α be the level α $(0<\alpha<1)$ point of χ_1^2 (i.e., $P[\chi_1^2 > x_\alpha] = \alpha$). In Fig. 1 we plot the graphs of $f(t)$ for $\alpha = 0.05$, 0.025, 0.01. From Fig. 1 we can design the third-order asymptotically most powerful test at each given t $(\varepsilon = I^{-1/2}t)$ and level $x = x_\alpha$. Conversely, for a given test $(\in\mathscr{S})$ with the coefficient $a_1 = \tilde{a}_1$ and level $x = x_\alpha$ we can find a value of t satisfying $I\tilde{a}_1 = f(t)$.

Now let us see the modification procedure $T \to T^{**} = (1 + \rho^*/c_n^2)$ $h(\hat{\theta}_{\mathrm{ML}})\,T$ described in Section 3 for the four tests LR, W, MW, and R. Their Bartlett adjustment factors ρ^* and the derivatives $h' = h'(\theta_0)$ and $h'' = h''(\theta_0)$ of their transformations are given by Table I.

More concretely we give two examples, I and II:

(I) Let $\{X_t\}$ be a Gaussian ARMA process with mean zero. Suppose that $\{X_t\}$ has the spectral density

$$f_{\theta_0}(\lambda) = \frac{\sigma^2}{2\pi} \frac{\prod_{k=1}^{q} (1 - \psi_k e^{i\lambda})(1 - \psi_k e^{-i\lambda})}{\prod_{k=1}^{p} (1 - \rho_k e^{i\lambda})(1 - \rho_k e^{-i\lambda})}, \tag{4.4}$$

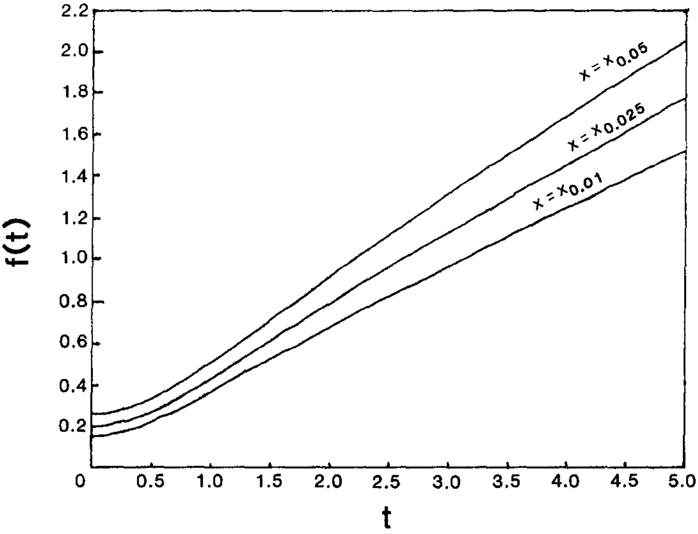

FIG. 1. The graphs of $f(t)$.

where $\psi_1, \ldots, \psi_q, \rho_1, \ldots, \rho_p$ are real numbers such that $|\psi_j| < 1$, $j = 1, \ldots, q$, $|\rho_j| < 1$, $j = 1, \ldots, p$. We also assume that $f_\theta(\lambda)$ is continuously five times differentiable with respect to θ. Then the quantities I, J, K, etc., are expressible in terms of the spectral density. For example,

$$I(\theta_0) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial \theta} \log f_{\theta_0}(\lambda) \right\}^2 d\lambda,$$

$$K(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial \theta} f_{\theta_0}(\lambda) \right\}^3 f_{\theta_0}(\lambda)^{-3} \, d\lambda$$

(see Taniguchi [12] concerning the other quantities). We can easily find the explicit forms of $h(\theta)$ for the W, MW, and R tests (see Table II).

Turning to the evaluation of Bartlett's adjustment factor ρ^* of $T^* = h(\hat{\theta}_{\mathrm{ML}})\, T$, we restrict ourselves to the case where the process $\{X_t\}$ is an $AR(1)$ or $MA(1)$ because calculation of Δ is very troublesome. Suppose that $\{X_t\}$ has the $AR(1)$ spectral density

$$f_{\theta_0}(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 - \rho e^{i\lambda}|^2}. \tag{4.5}$$

Then we can evaluate ρ^* for LR, W, MW, and R as shown in Table III.

TABLE I

Table of h', h'', and ρ^*

	LR	W	MW	R
h'	0	$-(3J+K)/3I$	$(3J+2K)/3I$	$-K/3I$
h''	0	$-\{12M+18N+8L$ $+3H\}/6I$ $+\{27J^2+20JK$ $+4K^2\}/6I^2$	$\dfrac{12N+4L+3H}{6I}$ $+\dfrac{J^2}{2I^2}$	$\dfrac{K^2-2KJ}{6I^2}$ $-\dfrac{H}{6I}$
ρ^*	$-\dfrac{\tilde{M}}{4I^2}+\dfrac{\tilde{N}}{2I^2}$ $+\dfrac{H}{4I^2}-\dfrac{5K^2}{12I^3}$	$\dfrac{\Delta}{I}+\dfrac{\tilde{N}}{I^2}$ $+\dfrac{H}{4I^2}-\dfrac{5K^2}{12I^3}$	$\dfrac{\Delta}{I}+\dfrac{\tilde{N}}{I^2}$ $+\dfrac{H}{4I^2}-\dfrac{5K^2}{12I^3}$	$-\dfrac{\Delta}{I}+\dfrac{H}{4I^2}$ $-\dfrac{5K^2}{12I^3}$

TABLE II

Table of $h(\theta)$ for the Tests W, MW, and R when $\{X_t\}$ Has the Spectral Density (4.4).

Test	Unknown parameter	Transformation $h(\theta)$
W	$\theta_0=\sigma^2$	$1+\dfrac{4}{3\sigma^2}(\theta-\sigma^2)+\dfrac{1}{6\sigma^4}(\theta-\sigma^2)^2$
	$\theta_0=\psi_m$	$1-\dfrac{2\psi_m}{1-\psi_m^2}(\theta-\psi_m)+\dfrac{3\psi_m^2-5}{2(1-\psi_m^2)^2}(\theta-\psi_m)^2$
	$\theta_0=\rho_m$	$1-\dfrac{(\theta-\rho_m)^2}{2(1-\rho_m^2)}$
MW	$\theta_0=\sigma^2$	$1-\dfrac{2}{3\sigma^2}(\theta-\sigma^2)+\dfrac{1}{2\sigma^4}(\theta-\sigma^2)^2$
	$\theta_0=\psi_m$	$1+\dfrac{\psi_m^2-3}{2(1-\psi_m^2)^2}(\theta-\psi_m)^2$
	$\theta_0=\rho_m$	$1+\dfrac{2\rho_m}{1-\rho_m^2}(\theta-\rho_m)+\dfrac{7\rho_m^2+1}{2(1-\rho_m^2)^2}(\theta-\rho_m)^2$
R	$\theta_0=\sigma^2$	$1-\dfrac{2}{3\sigma^2}(\theta-\sigma^2)+\dfrac{1}{2\sigma^4}(\theta-\sigma^2)^2$
	$\theta_0=\psi_m$	$1+\dfrac{2\psi_m}{1-\psi_m^2}(\theta-\psi_m)+\dfrac{7\psi_m^2-3}{2(1-\psi_m^2)^2}(\theta-\psi_m)^2$
	$\theta_0=\rho_m$	$1-\dfrac{2\rho_m}{1-\rho_m^2}(\theta-\rho_m)-\dfrac{3}{2(1-\rho_m^2)}(\theta-\rho_m)^2$

TABLE III

Table of ρ^* for the Tests LR, W, MW, and R when $\{X_t\}$ Has the Spectral Density (4.5)

Test T	Unknown parameter	$\rho^* = \rho^*(\theta_0)$ for $T^* = h(\hat{\theta}_{\mathrm{ML}})\, T$
LR	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \rho$	2
W	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \rho$	$\{5\rho^2 - 1\}/\{2(1-\rho^2)\}$
MW	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \rho$	$\{5\rho^2 - 1\}/\{2(1-\rho^2)\}$
R	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \rho$	$\{11 - 15\rho^2\}/\{2((1-\rho^2)\}$

Next we assume that $\{X_t\}$ has the $MA(1)$ spectral density

$$f_{\theta_0}(\lambda) = \frac{\sigma^2}{2\pi} \, |1 - \psi e^{i\lambda}|^2. \tag{4.6}$$

Similarly we have Table IV.

In Tables III and IV we observe that all the ρ^* for $\theta_0 = \sigma^2$ are independent of σ^2 (i.e., $\rho^* = -\frac{1}{3}$). For the autoregressive coefficient ρ in Table III it may be noted that ρ^* for the LR is also independent of ρ (i.e., $\rho^* = 2$).

TABLE IV

Table of ρ^* for the Tests LR, W, MW, and R when $\{X_t\}$ Has the Spectral Density (4.6)

Test T	Unknown parameter	$\rho^* = \rho^*(\theta_0)$ for $T^* = h(\hat{\theta}_{\mathrm{ML}})\, T$
LR	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \psi$	$-(1+2\psi^2)/(1-\psi^2)$
W	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \psi$	$-\{7\psi^2 + 9\}/\{2(1-\psi^2)\}$
MW	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \psi$	$-\{7\psi^2 + 9\}/\{2(1-\psi^2)\}$
R	$\theta_0 = \sigma^2$	$-1/3$
	$\theta_0 = \psi$	$\{11 - 3\psi^2\}/\{2(1-\psi^2)\}$

TABLE V

Table of h', h'', and ρ^* for the Model (4.7)

	LR	W	MW	R
h'	0	$-1/\theta$	$1/\theta$	0
h''	0	$5/2\theta^2$	$1/2\theta^2$	0
ρ^*	0	0	0	0

(II) Consider the nonlinear regression model

$$X_t = \alpha + \beta\theta^2 \cos(t-1)\,\lambda + u_t, \qquad t = 1, \ldots, n, \tag{4.7}$$

where θ is an unknown parameter satisfying $0 < \theta < \infty$, while α, β, and $\lambda = 2\pi l/n$ (l an integer) are known parameters, $\{u_t\}$ is a sequence of i.i.d. $N(0, \sigma^2)$ random variables. Then it follows that

$$I = 2\beta^2\theta^2/\sigma^2, \qquad J = 2\beta^2\theta/\sigma^2, \qquad M = 2\beta^2/\sigma^2,$$
$$K = N = L = H = \Delta = 0. \tag{4.8}$$

For our model (4.7) we can also calculate the Bartlett adjustment factor ρ^* and the derivatives h' and h'' for the four tests LR, W, MW, and R. From Table I and (4.8) the results are given by Table V.

Here it should be noted that $\tilde{M} = M - J^2/I = 0$. Recalling Theorem 3, we can see that, for any $T \in \mathscr{S}$, the local powers of all the modified tests T^{**} are identical up to third order.

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